

BLOW-UP PROBLEM FOR SEMILINEAR HEAT EQUATION WITH NONLINEAR NONLOCAL NEUMANN BOUNDARY CONDITION

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ABSTRACT. In this paper, we consider a semilinear parabolic equation with nonlinear nonlocal Neumann boundary condition and nonnegative initial datum. We first prove global existence results. We then give some criteria on this problem which determine whether the solutions blow up in finite time for large or for all nontrivial initial data. Finally, we show that under certain conditions blow-up occurs only on the boundary.

1. INTRODUCTION

In this paper we consider the initial boundary value problem for the following semilinear parabolic equation

$$u_t = \Delta u - c(x, t)u^p, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

with nonlinear nonlocal boundary condition

$$\frac{\partial u(x, t)}{\partial \nu} = \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad x \in \partial\Omega, \quad t \geq 0, \quad (1.2)$$

and initial datum

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.3)$$

where $p > 0$, $l > 0$, Ω is a bounded domain in \mathbb{R}^n for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is unit outward normal on $\partial\Omega$.

Throughout this paper we suppose that the functions $c(x, t)$, $k(x, y, t)$ and $u_0(x)$ satisfy the following conditions:

$$c(x, t) \in C_{loc}^{\alpha}(\overline{\Omega} \times [0, +\infty)), \quad 0 < \alpha < 1, \quad c(x, t) \geq 0;$$

$$k(x, y, t) \in C(\partial\Omega \times \overline{\Omega} \times [0, +\infty)), \quad k(x, y, t) \geq 0;$$

$$u_0(x) \in C^1(\overline{\Omega}), \quad u_0(x) \geq 0 \text{ in } \Omega, \quad \frac{\partial u_0(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0)u_0^l(y) dy \text{ on } \partial\Omega.$$

Many authors have studied blow-up problem for parabolic equations and systems with nonlocal boundary conditions (see, for example, [1]–[20] and the references therein). In particular, the initial boundary value problem for equation (1.1) with nonlinear nonlocal boundary condition

$$u(x, t) = \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad x \in \partial\Omega, \quad t > 0,$$

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was considered for $c(x, t) \leq 0$ and $c(x, t) \geq 0$ in [9] and [11], respectively. The problem (1.1)–(1.3) with $c(x, t) \leq 0$ was investigated in [8] and closed problem was analyzed in [14].

Local existence theorem, comparison and uniqueness results for problem (1.1)–(1.3) have been established in [21].

In this paper we obtain necessary and sufficient conditions for the existence of global solutions as well as for a blow-up in finite time of solutions for problem (1.1)–(1.3). Our global existence and blow-up results depend on the behavior of the functions $c(x, t)$ and $k(x, y, t)$ as $t \rightarrow \infty$.

This paper is organized as follows. The global existence theorem for any initial data and blow-up in finite time of solutions for large initial data are proved in section 2. In section 3 we present finite time blow-up of all nontrivial solutions as well as the existence of global solutions for small initial data. Finally, in section 4 we show that under certain conditions blow-up occurs only on the boundary.

2. GLOBAL EXISTENCE

Let $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$, $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

Definition 2.1. We say that a nonnegative function $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution of (1.1)–(1.3) in Q_T if

$$u_t \geq \Delta u - c(x, t)u^p, \quad (x, t) \in Q_T, \quad (2.1)$$

$$\frac{\partial u(x, t)}{\partial \nu} \geq \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad x \in \partial\Omega, \quad 0 \leq t < T, \quad (2.2)$$

$$u(x, 0) \geq u_0(x), \quad x \in \Omega, \quad (2.3)$$

and $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of (1.1)–(1.3) in Q_T if $u \geq 0$ and it satisfies (2.1)–(2.3) in the reverse order. We say that $u(x, t)$ is a solution of problem (1.1)–(1.3) in Q_T if $u(x, t)$ is both a subsolution and a supersolution of (1.1)–(1.3) in Q_T .

To prove the main results we use the positiveness of a solution and the comparison principle which have been proved in [21].

Theorem 2.2. *Let u_0 is a nontrivial function in Ω , $p \geq 1$ or $c(x, t) \equiv 0$. Suppose u is a solution of (1.1)–(1.3) in Q_T . Then $u > 0$ in $Q_T \cup S_T$.*

Theorem 2.3. *Let \bar{u} and \underline{u} be a supersolution and a subsolution of problem (1.1)–(1.3) in Q_T , respectively. Suppose that $\underline{u}(x, t) > 0$ or $\bar{u}(x, t) > 0$ in $Q_T \cup \Gamma_T$ if $l < 1$. Then $\bar{u}(x, t) \geq \underline{u}(x, t)$ in $Q_T \cup \Gamma_T$.*

The proof of a global existence result relies on the continuation principle and the construction of a supersolution. We suppose that

$$c(x, t) > 0, \quad x \in \overline{\Omega}, \quad t \geq 0. \quad (2.4)$$

Theorem 2.4. *Let $l \leq 1$ or $1 < l < p$ and (2.4) hold. Then problem (1.1)–(1.3) has a global solution for any initial datum.*

Proof. In order to prove the existence of global solutions we construct a suitable explicit supersolution of (1.1)–(1.3) in Q_T for any positive T . Suppose at first that $l \leq 1$. Since $k(x, y, t)$ is a continuous function there exists a constant $K > 0$ such that

$$k(x, y, t) \leq K \quad (2.5)$$

in $\partial\Omega \times Q_T$. Let λ_1 be the first eigenvalue of the following problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, & x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega, \end{cases}$$

and $\varphi(x)$ be the corresponding eigenfunction with $\sup_{\Omega} \varphi(x) = 1$. It is well known $\varphi(x) > 0$ in Ω and $\max_{\partial\Omega} \partial\varphi(x)/\partial\nu < 0$.

Now we show that

$$\bar{u}(x, t) = \frac{C \exp(\mu t)}{a\varphi(x) + 1}$$

is a supersolution of (1.1)–(1.3) in Q_T , where constants C, μ and a are chosen to satisfy the following inequalities:

$$a \geq \max \left\{ K \int_{\Omega} \frac{dy}{(\varphi(y) + 1)^l} \max_{\partial\Omega} \left(-\frac{\partial\varphi}{\partial\nu} \right)^{-1}, 1 \right\},$$

$$C \geq \max \{ \sup_{\Omega} (a\varphi(x) + 1) u_0(x), 1 \}, \quad \mu \geq \lambda_1 + 2a^2 \sup_{\Omega} \frac{|\nabla\varphi|^2}{(a\varphi(x) + 1)^2}.$$

Indeed, it is easy to check that

$$\bar{u}_t - \Delta\bar{u} + c(x, t)\bar{u}^p \geq \left(\mu - \frac{a\lambda_1\varphi}{(a\varphi(x) + 1)^2} - 2a^2 \sup_{\Omega} \frac{|\nabla\varphi|^2}{(a\varphi(x) + 1)^2} \right) \bar{u} \geq 0 \quad (2.6)$$

for $(x, t) \in Q_T$,

$$\begin{aligned} \frac{\partial\bar{u}}{\partial\nu} &= aC \exp(\mu t) \left(-\frac{\partial\varphi}{\partial\nu} \right) \geq KC^l \exp(l\mu t) \int_{\Omega} \frac{dy}{(\varphi(y) + 1)^l} \\ &\geq \int_{\Omega} k(x, y, t) \bar{u}^l(y, t) dy \end{aligned} \quad (2.7)$$

for $(x, t) \in S_T$ and

$$\bar{u}(x, 0) \geq u_0(x) \quad (2.8)$$

for $x \in \Omega$. It follows from (2.6)–(2.8) that problem (1.1)–(1.3) has a global solution for any initial datum.

Suppose now that $1 < l < p$ and (2.4) holds. By (2.4) we have $c(x, t) \geq \underline{c}$ in Q_T , where \underline{c} is some positive constant.

To construct a supersolution we use the change of variables in a neighborhood of $\partial\Omega$ as in [22]. Let \bar{x} be a point in $\partial\Omega$. We denote by $\hat{n}(\bar{x})$ the inner unit normal to $\partial\Omega$ at the point \bar{x} . Since $\partial\Omega$ is smooth it is well known that there exists $\delta > 0$ such that the mapping $\psi : \partial\Omega \times [0, \delta] \rightarrow \mathbb{R}^n$ given by $\psi(\bar{x}, s) = \bar{x} + s\hat{n}(\bar{x})$ defines new coordinates (\bar{x}, s) in a neighborhood of $\partial\Omega$ in $\bar{\Omega}$. A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\bar{x}, s) = g(s)$, which is independent of the variable \bar{x} , evaluated at a point (\bar{x}, s) is given by

$$\Delta g(\bar{x}, s) = \frac{\partial^2 g}{\partial s^2}(\bar{x}, s) - \sum_{j=1}^{n-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \frac{\partial g}{\partial s}(\bar{x}, s), \quad (2.9)$$

where $H_j(\bar{x})$ for $j = 1, \dots, n-1$, denotes the principal curvatures of $\partial\Omega$ at \bar{x} .

For points in $Q_{\delta, T} = \partial\Omega \times [0, \delta] \times [0, T]$ of coordinates (\bar{x}, s, t) define

$$\bar{u}(\bar{x}, s, t) = [(\alpha s + \varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta}{\gamma}} + A, \quad (2.10)$$

where $\alpha > 0$, $0 < \varepsilon < \omega < \alpha\delta$, $\max\{1/l, 2/(p-1)\} < \beta < 2/(l-1)$, $0 < \gamma < \beta/2$, $A \geq \sup_{\Omega} u_0(x)$, $\sigma_+ = \max\{\sigma, 0\}$. For points in $\overline{Q_T} \setminus Q_{\delta,T}$ we put $\overline{u}(\overline{x}, s, t) = A$. It has been showed in [11] that

$$\overline{u}_t - \Delta \overline{u} + c(x, t) \overline{u}^p \geq 0, \quad (x, t) \in Q_T$$

for small ε and large A .

Now we show that

$$\frac{\partial \overline{u}}{\partial \nu}(\overline{x}, 0, t) \geq \int_{\Omega} k(x, y, t) \overline{u}^l(\overline{x}, s, t) dy, \quad (x, t) \in Q_T \quad (2.11)$$

for a suitable choice of ε . To estimate the integral I in the right hand side of (2.11) we shall use the change of variables in a neighborhood of $\partial\Omega$. Let

$$\overline{J} = \sup_{0 < s < \delta} \int_{\partial\Omega} |J(\overline{y}, s)| d\overline{y},$$

where $J(\overline{y}, s)$ is Jacobian of the change of variables. Then we have

$$\begin{aligned} I &\leq 2^{l-1} K \int_{\Omega} [(\alpha s + \varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta l}{\gamma}} dy + 2^{l-1} K A^l |\Omega| \\ &\leq 2^{l-1} K \overline{J} \int_0^{(\omega - \varepsilon)/\alpha} [(\alpha s + \varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta l}{\gamma}} ds + 2^{l-1} K A^l |\Omega| \\ &\leq \frac{2^{l-1} K \overline{J}}{\alpha(\beta l - 1)} [\varepsilon^{-(\beta l - 1)} - \omega^{-(\beta l - 1)}] + 2^{l-1} K A^l |\Omega|, \end{aligned}$$

where K was defined in (2.5), $|\Omega|$ is Lebesgue measure of Ω . On the other hand, since

$$\frac{\partial \overline{u}}{\partial \nu}(\overline{x}, 0, t) = -\frac{\partial \overline{u}}{\partial s}(\overline{x}, 0, t) = \alpha \beta \varepsilon^{-\gamma-1} [\varepsilon^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta-\gamma}{\gamma}},$$

the inequality (2.11) holds if ε is small enough and hence by Theorem 2.3 we get

$$u(x, t) \leq \overline{u}(\overline{x}, s, t) \text{ in } \overline{Q_T}.$$

□

Remark 2.5. Let

$$\underline{\lambda} = \frac{\inf_{\Omega \times (0, +\infty)} c(x, t)}{\sup_{\partial\Omega \times \Omega \times (0, +\infty)} k(x, y, t)}.$$

Note that under $\beta = 2/(l-1)$ and a suitable choice of α in (2.10) the same proof holds if $l = p > 1$ and $\underline{\lambda}$ is large enough and consequently a solution of problem (1.1)–(1.3) is global.

Now we shall prove finite time blow-up result. We suppose that

$$k(x, y, t_0) > 0, \quad x \in \partial\Omega, \quad y \in \partial\Omega. \quad (2.12)$$

Theorem 2.6. *Let $l > \max\{1, p\}$ and (2.12) hold with $t_0 \geq 0$ if $p \leq 1$ and with $t_0 = 0$ if $p > 1$. Then there exist solutions of (1.1)–(1.3) with finite time blow-up.*

Proof. At first we suppose that $p \leq 1$, $l > 1$ and (2.12) holds with $t_0 \geq 0$. To prove the theorem we construct a subsolution of an auxiliary problem which blows up in finite time. First of all we get a lower bound for solutions of (1.1)–(1.3) with positive initial data. We denote

$$\overline{c}(t) = \sup_{\Omega} c(x, t). \quad (2.13)$$

It is not difficult to check that

$$w(t) = \begin{cases} \left[A^{1-p} - (1-p) \int_0^t \bar{c}(\tau) d\tau \right]^{1/(1-p)} & \text{for } 0 < p < 1, \\ A \exp \left[- \int_0^t \bar{c}(\tau) d\tau \right] & \text{for } p = 1 \end{cases}$$

is a subsolution of (1.1)–(1.3) in Q_T for any $T > 0$ if

$$u_0(x) \geq A > 0. \quad (2.14)$$

Then by Theorem 2.3 we have

$$u(x, t) \geq w(t) \text{ for } x \in \bar{\Omega} \text{ and } t \geq 0. \quad (2.15)$$

Consider the change of variables in a neighborhood of $\partial\Omega$ as in Theorem 2.4. Set $\Omega_\gamma = \{(\bar{x}, s) : \bar{x} \in \partial\Omega, 0 < s < \gamma\}$. By (2.12) we have

$$k(x, y, t) \geq k_1, \quad x \in \partial\Omega, y \in \Omega_\gamma, t_0 < t < t_1 \quad (2.16)$$

for some positive k_1 , γ and $t_1 > t_0$.

Let us consider the following initial boundary value problem:

$$\begin{cases} v_t = \Delta v - c(x, t)v^p & \text{for } x \in \Omega_\gamma, \quad t_0 < t < t_2, \\ \frac{\partial v(x, t)}{\partial \nu} = \int_{\Omega_\gamma} k(x, y, t)v^l(y, t) dy & \text{for } x \in \partial\Omega, \quad t_0 < t < t_2, \\ v(x, t) = u(x, t) & \text{for } x \in \partial\Omega_\gamma \setminus \partial\Omega, \quad t_0 < t < t_2, \\ v(x, t_0) = u(x, t_0) & \text{for } x \in \Omega_\gamma, \end{cases} \quad (2.17)$$

where ν is unit outward normal on $\partial\Omega$, $u(x, t)$ is a solution of (1.1)–(1.3), $t_2 \in (t_0, t_1)$ and will be chosen later. We can define the notions of a supersolution and a subsolution of (2.17) in a similar way as in Definition 2.1. We shall use a comparison principle for a subsolution and a supersolution of (2.17) which can be proved analogously to Theorem 2.3. It is easy to see that $u(x, t)$ is a supersolution of (2.17) in $Q(\gamma, t_0, t_2) = \Omega_\gamma \times (t_0, t_2)$.

We define

$$\psi(s, t) = (t_2 + s - t)^{-\sigma}, \quad (2.18)$$

where $\sigma > 2/(l-1)$ and show that $\psi(s, t)$ is a subsolution of (2.17) in $Q(\gamma, t_0, t_2)$ under suitable choice of t_2 and γ . It is obvious, $\psi(0, t) \rightarrow \infty$ as $t \rightarrow t_2$.

For $0 < s < \gamma$ and small γ we have

$$\left| \sum_{j=1}^{n-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \right| \leq C. \quad (2.19)$$

Using (2.9), (2.18), (2.19) we find that

$$\begin{aligned} -\psi_t + \Delta\psi - c(x, t)\psi^p &\geq (t_2 + s - t)^{-\sigma-2} \{ \sigma(\sigma+1) - \sigma(C+1)(t_2 - t_0 + \gamma) \\ &\quad - \sup_{(t_0, t_2)} \bar{c}(t)(t_2 - t_0 + \gamma)^{\sigma+2-\sigma p} \} \geq 0 \end{aligned}$$

in $Q(\gamma, t_0, t_2)$ if we take γ and $t_2 - t_0$ small enough. Now we prove that

$$\frac{\partial \psi}{\partial \nu}(0, t) \leq \int_{\Omega_\gamma} k(x, y, t)\psi^l(s, t) dy \text{ for } x \in \partial\Omega, t_0 < t < t_2.$$

To do this, we use the change of variables in a neighborhood of $\partial\Omega$. Let

$$\underline{J} = \inf_{0 < s < \gamma} \int_{\partial\Omega} |J(\bar{y}, s)| d\bar{y},$$

where $J(\overline{y}, s)$ is Jacobian of the change of variables. By virtue of (2.16), (2.18) we have

$$\begin{aligned} \frac{\partial \psi}{\partial \nu}(0, t) &= \int_{\Omega_\gamma} k(x, y, t) \psi^l(s, t) dy \\ &\leq \sigma(t_2 - t)^{-\sigma-1} - k_1 \underline{J} \int_0^\gamma (t_2 + s - t)^{-\sigma l} ds \\ &\leq \sigma(t_2 - t)^{-\sigma-1} - k_1 \underline{J} \frac{(t_2 - t)^{-\sigma l + 1}}{\sigma l - 1} \left[1 - \left(\frac{t_2}{t_2 + \gamma} \right)^{\sigma l - 1} \right] \leq 0 \end{aligned}$$

for $x \in \partial\Omega$, $t_0 < t < t_2$ and small enough $t_2 - t_0$.

We suppose now that

$$\gamma < t_2 - t_0, \quad (2.20)$$

$$A \geq \left[(1-p) \int_0^{t_1} \overline{c}(\tau) d\tau + \gamma^{-\sigma(1-p)} \right]^{1/(1-p)} \quad \text{for } 0 < p < 1, \quad (2.21)$$

$$A \geq \gamma^{-\sigma} \exp \left[\int_0^{t_1} \overline{c}(\tau) \right] \quad \text{for } p = 1. \quad (2.22)$$

Due to (2.14), (2.15), (2.20) – (2.22) we have

$$\psi(s, t) \leq u(x, t) \quad \text{for } x \in \Omega_\gamma, t = t_0 \quad \text{and } x \in \partial\Omega_\gamma \setminus \partial\Omega, \quad t_0 \leq t \leq t_2.$$

Comparing $u(x, t)$ and $\psi(s, t)$ in $Q(\gamma, t_0, t_2)$ we prove the theorem for $p \leq 1$, $l > 1$.

Let $l > p > 1$ and (2.12) hold with $t_0 = 0$. We denote $c_1 = \sup_{Q_{t_1}} c(x, t)$ and suppose that

$$\max \left\{ \frac{1}{p-1}, \frac{2}{l-1} \right\} < \sigma < \frac{2}{p-1}, \quad u_0(x) \geq \max \left\{ [t_2(p-1)c_1]^{-\frac{1}{p-1}}, t_2^{-\sigma} \right\},$$

where $t_2 \in (0, t_1)$ and will be chosen later. It is not difficult to check that

$$w(t) = [(p-1)c_1(t+t_2)]^{-\frac{1}{p-1}}$$

is a subsolution of (1.1)–(1.3) in Q_{t_2} . Then by Theorem 2.3 we have

$$w(t) \leq u(x, t) \quad \text{for } x \in \overline{\Omega} \quad \text{and } 0 \leq t \leq t_2.$$

In the same way as in a previous case we can show that $\psi(s, t)$ is a subsolution of (2.17) in $Q(\gamma, t_0, t_2)$ with $t_0 = 0$ for small values of γ and

$$t_2 \leq \min \left\{ t_1, \frac{\gamma^{\sigma(p-1)}}{2(p-1)c_1} \right\}.$$

□

Remark 2.7. We put

$$\overline{\lambda} = \frac{\sup_{\partial\Omega} c(x, 0)}{\inf_{\partial\Omega \times \partial\Omega} k(x, y, 0)}$$

and consider

$$\psi(s, t) = (t_2 + \omega s - t)^{-2/(p-1)}, \quad \omega > 0 \quad (2.23)$$

instead of (2.18). Under a suitable choice of ω in (2.23) the same proof holds for $l = p > 1$ if $\overline{\lambda}$ is small enough and hence there exist solutions of (1.1)–(1.3) with finite time blow-up.

3. BLOW-UP OF ALL NONTRIVIAL SOLUTIONS

In this section we find the conditions which guarantee blow-up in finite time of all nontrivial solutions of (1.1)–(1.3).

First we prove that for $p < 1$ and $l > 1$ no blow-up of all nontrivial solutions of (1.1)–(1.3) if

$$\inf_{\Omega} c(x, 0) > 0. \quad (3.1)$$

Theorem 3.1. *Let $p < 1$, $l > 1$ and (3.1) hold. Then problem (1.1)–(1.3) has global solutions for small initial data.*

Proof. Thanks to the assumptions of the theorem we have $c(x, t) \geq c_0$ and $k(x, y, t) \leq K$ in Q_τ and $\partial\Omega \times Q_\tau$, respectively, where c_0 , K and τ are some positive constants.

Let $\psi(x)$ be a positive solution of the following problem

$$\Delta\psi = 1, \quad x \in \Omega; \quad \frac{\partial\psi(x)}{\partial\nu} = \frac{|\Omega|}{|\partial\Omega|}, \quad x \in \partial\Omega. \quad (3.2)$$

We put

$$b = \inf_{\Omega} \psi(x) \quad (3.3)$$

and suppose that $f(t)$ is a solution of the following equation

$$f'(t) = \frac{f(t)}{b} - c_0 b^{p-1} f^p(t).$$

Then $f(t)$ can be written in an explicit form

$$f(t) = \exp(t/b) \left\{ f^{1-p}(0) - c_0 b^p (1 - \exp[(p-1)t/b]) \right\}_+^{1/(1-p)}.$$

We assume that

$$0 < f(0) < \{c_0 b^p (1 - \exp[(p-1)\tau/b])\}^{1/(1-p)}.$$

Then $f(t) \equiv 0$ for $t \geq \tau$.

To prove the theorem we construct a supersolution of (1.1)–(1.3) in such a form that $v(x, t) = \psi(x)f(t)$. It is not difficult to check that

$$v_t - \Delta v + c(x, t)v^p \geq 0 \quad (3.4)$$

for $x \in \Omega$, $t > 0$. Now we show that

$$\frac{\partial v}{\partial\nu}(x, t) \geq \int_{\Omega} k(x, y, t)v^l(y, t) dy \quad x \in \partial\Omega, \quad t > 0 \quad (3.5)$$

for a suitable choice of $f(0)$. Indeed,

$$\frac{\partial v}{\partial\nu}(x, t) = \frac{|\Omega|}{|\partial\Omega|} f(t) \geq \int_{\Omega} k(x, y, t)\psi^l(y)f^l(t) dy = \int_{\Omega} k(x, y, t)v^l(y, t) dy, \quad x \in \partial\Omega, \quad t > 0$$

for small values of $f(0)$. By (3.4), (3.5) we conclude that $v(x, t)$ is a supersolution of (1.1)–(1.3) in Q_T for any $T > 0$ if

$$u_0(x) \leq \psi(x)f(0), \quad x \in \Omega.$$

Now Theorem 2.3 guarantees the existence of global solutions of (1.1)–(1.3) for small initial data. \square

The following two statements deal with the case $p = 1$, $l > 1$. Let us introduce the notations

$$\underline{c}(t) = \inf_{\Omega} c(x, t), \quad \overline{k}_c(t) = \sup_{\partial\Omega \times \Omega} k(x, y, t) \exp \left\{ -(l-1) \int_0^t \underline{c}(\tau) d\tau \right\},$$

$$\underline{k}_c(x, t) = \inf_{\Omega} k(x, y, t) \exp \left\{ -(l-1) \int_0^t \overline{c}(\tau) d\tau \right\},$$

where $\overline{c}(t)$ was defined in (2.13).

We prove that any nontrivial solution of (1.1)–(1.3) blows up in finite time if

$$\int_0^\infty \int_{\partial\Omega} \underline{k}_c(x, t) dS_x dt = \infty. \quad (3.6)$$

Conversely, problem (1.1)–(1.3) has bounded global solutions with small initial data, provided that

$$\int_0^\infty \overline{k}_c(t) dt < \infty, \quad (3.7)$$

and there exist positive constants α , t_0 and K such that $\alpha > t_0$ and

$$\int_{t-t_0}^t \frac{\overline{k}_c(\tau)}{\sqrt{t-\tau}} d\tau \leq K \text{ for any } t \geq \alpha. \quad (3.8)$$

Theorem 3.2. *Let $p = 1$, $l > 1$ and (3.6) hold. Then any nontrivial solution of (1.1)–(1.3) blows up at time $t^* \leq T$, where T satisfies the equality*

$$\int_0^T \int_{\partial\Omega} \underline{k}_c(x, t) dS_x dt = \frac{1}{(l-1)} \left\{ |\Omega| \int_{\Omega} u_0(y) dy \right\}^{-(l-1)}.$$

Proof. Let $v(x, t)$ be a solution of the following problem

$$v_t = \Delta v \text{ for } x \in \Omega, t > 0, \quad (3.9)$$

$$\frac{\partial v(x, t)}{\partial \nu} = \underline{k}_c(x, t) \int_{\Omega} v^l(y, t) dy \text{ for } x \in \partial\Omega, t > 0, \quad (3.10)$$

$$v(x, 0) = u_0(x) \text{ for } x \in \Omega, \quad (3.11)$$

By a direct computation we can check that

$$\underline{u}(x, t) = \exp \left(- \int_0^t \overline{c}(\tau) d\tau \right) v(x, t)$$

is a subsolution of (1.1)–(1.3) in Q_T for any $T > 0$. Then by Theorem 2.3 we have

$$\underline{u}(x, t) \leq u(x, t), (x, t) \in Q_T$$

for any $T > 0$. To prove the theorem we show that any nontrivial solution of (3.9)–(3.11) blows up in finite time. We set

$$V(t) = \int_{\Omega} v(x, t) dx.$$

Integrating (3.9) over Ω and using Green's identity and Jensen's inequality, we have

$$\begin{aligned} V'(t) &= \int_{\Omega} \Delta v(x, t) dx = \int_{\partial\Omega} \frac{\partial v(x, t)}{\partial \nu} dS_x = \int_{\partial\Omega} \underline{k}_c(x, t) dS_x \int_{\Omega} v^l(y, t) dy \\ &\geq |\Omega|^{1-l} \int_{\partial\Omega} \underline{k}_c(x, t) dS_x V^l(t). \end{aligned}$$

Integrating last inequality, we obtain the desired result due to (3.6). \square

Theorem 3.3. *Let $p = 1$, $l > 1$ and (3.7), (3.8) hold. Then problem (1.1)–(1.3) has bounded global solutions for small initial data.*

Proof. Let $w(x, t)$ be a solution of the following problem

$$\begin{cases} w_t = \Delta w & \text{for } x \in \Omega, t > 0, \\ \frac{\partial w(x, t)}{\partial \nu} = \bar{k}_c(t) \int_{\Omega} w^l(y, t) dy & \text{for } x \in \partial\Omega, t > 0, \\ w(x, 0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \quad (3.12)$$

By a direct computation we can check that

$$\bar{u}(x, t) = \exp\left(-\int_0^t \underline{c}(\tau) d\tau\right) w(x, t)$$

is a supersolution of (1.1)–(1.3) in Q_T for any $T > 0$. To prove the theorem we show the existence of global bounded solutions of (3.12). Let us consider the following auxiliary linear problem

$$\begin{cases} h_t = \Delta h, x \in \Omega, t > 0 \\ \frac{\partial h(x, t)}{\partial \nu} = \bar{k}_c(t), x \in \partial\Omega, t > 0, \\ h(x, 0) = h_0(x), x \in \Omega. \end{cases} \quad (3.13)$$

As it was proved in [8] any solution of (3.13) is a bounded function. Now we construct a supersolution of (3.12) in the following form $g(x, t) = ah(x, t)$, where a is some positive constant. It is obvious,

$$g_t = \Delta g, x \in \Omega, t > 0.$$

Moreover,

$$\frac{\partial g(x, t)}{\partial \nu} = a\bar{k}_c(t) \geq a^l \bar{k}_c(t) \int_{\Omega} h^l(y, t) dy = \bar{k}_c(t) \int_{\Omega} g^l(y, t) dy, x \in \partial\Omega, t > 0,$$

for small values of a . Then by a comparison principle for (3.12)

$$w(x, t) \leq g(x, t), (x, t) \in Q_T$$

for any $T > 0$ if $u_0(x) \leq ah_0(x)$, $x \in \Omega$. \square

Remark 3.4. By Theorem 3.2 and Theorem 3.3 the condition (3.7) is optimal for global existence of solutions of (1.1)–(1.3) with $c(x, t) = c(t)$ and $k(x, y, t) = k(t)$. Arguing in the same way as in the proof of Lemma 3.3 of [8] it is easy to show that (3.8) is optimal for the existence of nontrivial bounded global solutions of (1.1)–(1.3) with $c(x, t) = c(t)$ and $k(x, y, t) = k(t)$ under the condition

$$\int_0^\infty c(t) dt < \infty.$$

Now we prove finite time blow-up of all nontrivial solutions of (1.1)–(1.3) for $l > p > 1$. Let $m_0 = \inf\{\sup_{\Omega} \psi(x)\}$, where $\psi(x)$ was defined in (3.2). To formulate blow-up result we put

$$\underline{k}(t) = \inf_{\partial\Omega \times \Omega} k(x, y, t)$$

and suppose that

$$c(x, t) \leq c_1(t), c_1(t) \in C^1([t_0, \infty)), c_1(t) > 0 \text{ for } t \geq t_0, \quad (3.14)$$

where t_0 is some positive constant,

$$\liminf_{t \rightarrow \infty} \frac{c'_1(t)}{c_1(t)} > -\frac{p-1}{m_0} \quad (3.15)$$

and

$$\lim_{t \rightarrow \infty} \underline{k}(t)[c_1(t)]^{(1-l)/(p-1)} = \infty. \quad (3.16)$$

Theorem 3.5. *Let $l > p > 1$ and (3.14) – (3.15) hold. Then any nontrivial solution of (1.1)–(1.3) blows up in finite time.*

Proof. Let $u(x, t)$ be a nontrivial global solution of (1.1)–(1.3). Then by Theorem 2.2

$$u(x, t) > 0 \text{ for } x \in \overline{\Omega}, \ t > 0. \quad (3.17)$$

At first we get an universal lower bound for $u(x, t)$. From (3.15) we see that there exists a constant m satisfying $m > m_0$ and

$$\liminf_{t \rightarrow \infty} \frac{c'_1(t)}{c_1(t)} > -\frac{p-1}{m}. \quad (3.18)$$

Let us define $f(t)$ as a solution of the following equation

$$f'(t) = \frac{f(t)}{m} - m^{p-1}c_1(t)f^p(t), \ t \geq t_1 \geq t_0, \quad (3.19)$$

Then $f(t)$ can be written in an explicit form

$$f(t) = \exp(t/m) \left\{ [f(t_1) \exp(-t_1/m)]^{1-p} + (p-1)m^{p-1} \int_{t_1}^t \exp[(p-1)\tau/m] c_1(\tau) d\tau \right\}^{-\frac{1}{p-1}}. \quad (3.20)$$

We rewrite (3.20) as following

$$\left\{ \frac{f(t)}{[c_1(t)]^{-1/(p-1)}} \right\}^{p-1} = \frac{\exp[(p-1)t/m] c_1(t)}{[f(t_1) \exp(-t_1/m)]^{1-p} + (p-1)m^{p-1} \int_{t_1}^t \exp[(p-1)\tau/m] c_1(\tau) d\tau}. \quad (3.21)$$

We prove that right hand side I of (3.21) is bounded below by some positive constant. The numerator and the denominator of I tend to infinity as $t \rightarrow \infty$ by virtue of (3.15). Using (3.18) we can obtain that

$$\liminf_{t \rightarrow \infty} I \geq \liminf_{t \rightarrow \infty} \frac{\exp[(p-1)t/m] \{(p-1)c_1(t)/m + c'_1(t)\}}{(p-1)m^{p-1}c_1(t) \exp[(p-1)t/m]} > 0. \quad (3.22)$$

By (3.20) – (3.22) we conclude that

$$f(t) \geq d_1 [c_1(t)]^{-\frac{1}{p-1}}, \ t \geq t_1, \quad (3.23)$$

where $d_1 > 0$.

Let $\psi(x)$ satisfy (3.2) and

$$\sup_{\Omega} \psi(x) = m. \quad (3.24)$$

Now we define

$$\underline{u}(x, t) = \psi(x)f(t) \quad (3.25)$$

and show that $\underline{u}(x, t)$ is a subsolution of (1.1)–(1.3) in $\Omega \times (t_1, T)$ under suitable choice of t_1 and $T > t_1$. Due to (3.2), (3.19) we have

$$\underline{u}_t \leq \Delta \underline{u} - c(x, t)\underline{u}^p, \ x \in \Omega, \ t > t_1. \quad (3.26)$$

Using (3.2), (3.16), (3.23), (3.25) we find that

$$\begin{aligned} \frac{\partial \underline{u}}{\partial \nu}(x, t) &= \frac{|\Omega|}{|\partial \Omega|} f(t) \leq d_1^{l-1} [c_1(t)]^{-\frac{l-1}{p-1}} \underline{k}(t) f(t) \int_{\Omega} \psi^l(y) dy \\ &\leq \int_{\Omega} k(x, y, t) \underline{u}^l(y, t) dy, \quad x \in \partial \Omega, \quad t > t_1 \end{aligned} \quad (3.27)$$

for large values of t_1 . By (3.17), (3.23) – (3.27) and Theorem 2.3

$$u(x, t) \geq \underline{u}(x, t) \geq d_2 [c_1(t)]^{-\frac{1}{p-1}}, \quad (x, t) \in \Omega \times (t_1, T) \quad (3.28)$$

for some $d_2 > 0$ and any $T > t_1$ if

$$f(t_1) \leq \frac{\inf_{\Omega} u(x, t_1)}{m}.$$

We set

$$U(t) = \int_{\Omega} u(x, t) dx. \quad (3.29)$$

Integrating (1.1) over Ω and using (3.14), (3.16), (3.28), (3.29) and Green's identity, we have

$$\begin{aligned} U'(t) &= \int_{\Omega} (\Delta u(x, t) - c(x, t) u^p(x, t)) dx \geq \int_{\Omega} (|\partial \Omega| \underline{k}(t) u^l(x, t) - c_1(t) u^p(x, t)) dx \\ &\geq \frac{1}{2} |\partial \Omega| \underline{k}(t) \int_{\Omega} u^l(x, t) dx \geq \frac{1}{2} |\partial \Omega| d_2^{l-1} \underline{k}(t) [c_1(t)]^{-\frac{l-1}{p-1}} \int_{\Omega} u(x, t) dx \\ &= \xi(t) U(t), \end{aligned} \quad (3.30)$$

where $t \geq t_2$, t_2 is large enough and $\lim_{t \rightarrow \infty} \xi(t) dt = \infty$. Integrating (3.30) over (t_2, t) we find that

$$U(t) \geq U(t_2) \exp \left(\int_{t_2}^t \xi(\tau) d\tau \right). \quad (3.31)$$

Now we deduce lower bound for $\underline{k}(t)$. From (3.18) we conclude that

$$c_1(t) \geq c_1(t_3) \exp \left(-\frac{(p-1)t}{m} \right), \quad t \geq t_3 \quad (3.32)$$

for some $t_3 \geq t_2$. By (3.16), (3.32) we have

$$\underline{k}(t) = \gamma_1(t) [c_1(t)]^{(l-1)/(p-1)} \geq \gamma_2(t) \exp \left(-\frac{(l-1)t}{m} \right) \quad \text{for } t \geq t_3, \quad (3.33)$$

where $\lim_{t \rightarrow \infty} \gamma_i(t) = \infty$, $i = 1, 2$.

Let us change unknown function

$$w(x, t) = \exp \left(-\frac{t}{m} \right) u(x, t). \quad (3.34)$$

It is easy to check that $w(x, t)$ is a solution of the following problem

$$w_t = \Delta w - c(x, t) \exp \left(\frac{(p-1)t}{m} \right) w^p - \frac{1}{m} w, \quad x \in \Omega, \quad t > 0, \quad (3.35)$$

$$\frac{\partial w(x, t)}{\partial \nu} = \exp \left(\frac{(l-1)t}{m} \right) \int_{\Omega} k(x, y, t) w^l(y, t) dy, \quad x \in \partial \Omega, \quad t \geq 0, \quad (3.36)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega.$$

We put

$$W(t) = \int_{\Omega} w(x, t) dx. \quad (3.37)$$

From (3.29), (3.31), (3.34), (3.37) we conclude that

$$\lim_{t \rightarrow \infty} W(t) = \infty.$$

Integrating (3.35) over Ω and using (3.14), (3.16), (3.28), (3.30), (3.33), (3.36), (3.37), Green's identity and Jensen's inequality, we have

$$W'(t) \geq \sigma(t)W'(t) - \frac{1}{m}W(t) \text{ for } t \geq t_3,$$

where $\lim_{t \rightarrow \infty} \sigma(t) = \infty$. Hence $W(t)$ blows in finite time. \square

To prove the optimality of (3.16) for blow-up of any nontrivial solution of (1.1)–(1.3) we put

$$\bar{k}(t) = \sup_{\partial\Omega \times \Omega} k(x, y, t)$$

and assume that

$$c(x, t) \geq c_2(t) \text{ for } t \geq 0, \quad c_2(t) \in C([0, \infty)) \cap C^1([\sigma, \infty)), \quad c_2(t) > 0 \text{ for } t \geq \sigma, \quad (3.38)$$

$$\limsup_{t \rightarrow \infty} \frac{c_2'(t)}{c_2(t)} \leq 0, \quad (3.39)$$

$$\bar{k}(t) \leq K_c [c_2(t)]^{(l-1)/(p-1)}, \quad t \geq 0, \quad (3.40)$$

where σ and K_c are some positive constants.

Theorem 3.6. *Let $l > p > 1$ and (3.38) – (3.39) hold. Then problem (1.1)–(1.3) has global solutions for small initial data.*

Proof. To prove the theorem we construct a supersolution of (1.1)–(1.3) in Q_T for any $T > 0$. Let us define $g(t)$ as a positive solution of the following equation

$$g'(t) = \frac{g(t)}{b} - b^{p-1}c_2(t)g^p(t), \quad (3.41)$$

where b was defined in (3.3). Then $g(t)$ can be written in an explicit form

$$g(t) = \exp(t/b) \left\{ [g(0)]^{1-p} + (p-1)b^{p-1} \int_0^t \exp[(p-1)\tau/b] c_2(\tau) d\tau \right\}^{-1/(p-1)}. \quad (3.42)$$

We rewrite (3.42) as following

$$\left\{ g(t)[c_2(t)]^{1/(p-1)} \right\}^{p-1} = \frac{\exp[(p-1)t/b] c_2(t)}{[g(0)]^{1-p} + (p-1)b^{p-1} \int_0^t \exp[(p-1)\tau/b] c_2(\tau) d\tau}. \quad (3.43)$$

Defining the functions

$$\alpha(t) = \exp[(p-1)t/b] c_2(t),$$

$$\beta(t) = [g(0)]^{1-p} + (p-1)b^{p-1} \int_0^t \exp[(p-1)\tau/b] c_2(\tau) d\tau$$

and using Cauchy's mean value theorem and (3.39) for large values of a we obtain

$$\frac{\alpha(t)}{\beta(t)} - \frac{\alpha(a)}{\beta(a)} \leq \frac{\alpha(t) - \alpha(a)}{\beta(t) - \beta(a)} = \frac{\alpha'(\xi)}{\beta'(\xi)} = \frac{1}{b^p} + \frac{1}{(p-1)b^{p-1}} \frac{c_2'(\xi)}{c_2(\xi)} \leq \frac{2}{b^p}, \quad (3.44)$$

where $t > a$ and $\xi \in (a, t)$. From (3.43), (3.44) we deduce that

$$\left\{ g(t)[c_2(t)]^{1/(p-1)} \right\}^{p-1} \leq \frac{3}{b^p}, \quad t \geq 0 \quad (3.45)$$

for small values of $g(0)$.

Now we define

$$\bar{u}(x, t) = \psi(x)g(t) \quad (3.46)$$

and show that $\bar{u}(x, t)$ is a supesolution of (1.1)–(1.3) in Q_T for any $T > 0$ if initial data are small. By (3.2), (3.41) we have

$$\bar{u}_t - \Delta \bar{u} + c(x, t)\bar{u}^p \geq 0, \quad x \in \Omega, \quad t > 0. \quad (3.47)$$

We note that

$$\lim_{b \rightarrow \infty} \frac{m}{b} = 1, \quad (3.48)$$

where m was defined in (3.24). Using (3.2), (3.40), (3.45), (3.46), (3.48) we find that

$$\begin{aligned} \frac{\partial \bar{u}}{\partial \nu}(x, t) &= \frac{|\Omega|}{|\partial \Omega|} g(t) \geq K_c |\Omega| m^l \left[\frac{3}{b^p} \right]^{\frac{l-1}{p-1}} g(t) \\ &\geq K_c \left\{ g(t)[c_2(t)]^{1/(p-1)} \right\}^{l-1} g(t) \int_{\Omega} \psi^l(y) dy \geq \bar{k}(t) g^l(t) \int_{\Omega} \psi^l(y) dy \\ &\geq \int_{\Omega} k(x, y, t) \bar{u}^l(y, t) dy, \quad x \in \partial \Omega, \quad t > 0 \end{aligned} \quad (3.49)$$

for large values of b . Thus, by (3.47), (3.49) and Theorem 2.3 $\bar{u}(x, t)$ is a supersolution of (1.1)–(1.3) in Q_T for any $T > 0$ if

$$u_0(x) \leq g(0)\psi(x).$$

□

We shall write $h(x, t) \sim s(t)$ and $z(x, y, t) \sim s(t)$ as $t \rightarrow \infty$ if there exist positive constants β_i , ($i = \overline{1, 6}$) such that

$$\beta_1 h(x, t) \leq s(t) \leq \beta_2 h(x, t) \text{ for } x \in \Omega \text{ and } t \geq \beta_3$$

and

$$\beta_4 z(x, y, t) \leq s(t) \leq \beta_5 z(x, y, t) \text{ for } x \in \partial \Omega, \quad y \in \Omega \text{ and } t \geq \beta_6,$$

respectively.

Remark 3.7. By Theorem 3.5 and Theorem 3.6 the condition (3.16) is optimal in a certain sense for blow-up in finite time of any nontrivial solution of (1.1)–(1.3). In particular, let $c(x, t) \sim t^\alpha \ln^\beta t$ as $t \rightarrow \infty$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Then there exist global solutions of (1.1)–(1.3) for $k(x, y, t) \leq z(t)$, where $z(t) \sim \{t^\alpha \ln^\beta t\}^{(l-1)/(p-1)}$ as $t \rightarrow \infty$ and any nontrivial solution of (1.1)–(1.3) blows up in finite time for $k(x, y, t) \sim \gamma(t) \{t^\alpha \ln^\beta t\}^{(l-1)/(p-1)}$ as $t \rightarrow \infty$ if $\lim_{t \rightarrow \infty} \gamma(t) = \infty$.

4. BLOW-UP ON THE BOUNDARY

In this section we show that for problem (1.1)–(1.3) under certain conditions blow-up cannot occur at the interior domain.

Lemma 4.1. *Let $l > \max\{p, 1\}$, $\inf_{\partial\Omega \times Q_T} k(x, y, t) > 0$ and the solution $u(x, t)$ of (1.1)–(1.3) blows up in $t = T$. Then for $t \in [0, T)$*

$$\int_0^t \int_{\Omega} u^l(x, \tau) dx d\tau \leq s(T - t)^{-1/(l-1)}, \quad s > 0. \quad (4.1)$$

Proof. Integrating (1.1) over Q_t and using Green's identity, we have

$$\begin{aligned} \int_{\Omega} u(y, t) dy &= \int_{\Omega} u_0(y) dy + \int_0^t \int_{\partial\Omega} \int_{\Omega} k(\xi, y, \tau) u^l(y, \tau) dy dS_{\xi} d\tau \\ &\quad - \int_0^t \int_{\Omega} c(y, \tau) u^p(y, \tau) dy d\tau \geq \int_0^t \int_{\Omega} (k|\partial\Omega| u^l(y, \tau) - C u^p(y, \tau)) dy d\tau \\ &\geq \frac{k|\partial\Omega|}{2} \int_0^t \int_{\Omega} u^l(y, \tau) dy d\tau - M, \end{aligned} \quad (4.2)$$

where

$$k = \inf_{\partial\Omega \times Q_T} k(x, y, t), \quad C = \sup_{Q_T} c(x, t), \quad M = T|\Omega| \left\{ \frac{2C^{l/p}}{k|\partial\Omega|} \right\}^{\frac{p}{l-p}}.$$

Applying Hölder's inequality, we obtain

$$\int_{\Omega} u(y, t) dy \leq |\Omega|^{(l-1)/l} \left\{ \int_{\Omega} u^l(y, t) dy \right\}^{1/l}. \quad (4.3)$$

Let us introduce

$$J(t) = \int_0^t \int_{\Omega} u^l(x, \tau) dx d\tau.$$

Now from (4.2), (4.3) we have

$$(J'(t))^{1/l} \geq c_0 J(t) - M_1, \quad c_0 > 0, M_1 > 0.$$

We suppose there exists $t_0 \in (0, T)$ such that $J(t_0) = 2M_1/c_0$ since otherwise (4.1) holds. Then $J(t) \leq 2M_1/c_0$ for $0 \leq t \leq t_0$ and

$$J'(t) \geq \left(\frac{c_0}{2} J(t) \right)^l \quad \text{for } t \geq t_0. \quad (4.4)$$

Integrating (4.4) over $(t; T)$, we obtain (4.1). \square

Theorem 4.2. *Let the conditions of Lemma 4.1 hold. Then for problem (1.1)–(1.3) blow-up can occur only on the boundary.*

Proof. In the proof we shall use some arguments of [23], [24]. Let $G_N(x, y; t - \tau)$ be the Green function of the heat equation with homogeneous Neumann boundary condition. Then we have the representation formula:

$$\begin{aligned} u(x, t) &= \int_{\Omega} G_N(x, y; t) u_0(y) dy - \int_0^t \int_{\Omega} G_N(x, y; t - \tau) c(y, \tau) u^p(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\partial\Omega} G_N(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) u^l(y, \tau) dy dS_{\xi} d\tau \end{aligned} \quad (4.5)$$

for $(x, t) \in Q_T$. We now take an arbitrary $\Omega' \subset\subset \Omega$ with $\partial\Omega' \in C^2$ such that $\text{dist}(\partial\Omega, \Omega') = \varepsilon > 0$. It is well known (see, for example, [25], [26]) that

$$G_N(x, y; t - \tau) \geq 0, \quad x, y \in \Omega, \quad 0 \leq \tau < t < T, \quad (4.6)$$

$$\int_{\Omega} G_N(x, y; t - \tau) dy = 1, \quad x \in \Omega, \quad 0 \leq \tau < t < T. \quad (4.7)$$

$$0 \leq G_N(x, y; t - \tau) \leq c_{\varepsilon}, \quad x \in \Omega', \quad y \in \partial\Omega, \quad 0 < \tau < t < T, \quad (4.8)$$

where c_{ε} is a positive constant depending on ε . By (4.1), (4.5) – (4.8) we have

$$\begin{aligned} \sup_{\Omega'} u(x, t) &\leq \sup_{\Omega} u_0(x) + c_{\varepsilon} |\partial\Omega| \sup_{\partial\Omega \times Q_T} k(x, y, t) \int_0^t \int_{\Omega} u^l(y, \tau) dy d\tau \\ &\leq c_1 (T - t)^{-1/(l-1)}. \end{aligned}$$

As it is shown in [24], there exist a function $f(x) \in C^2(\overline{\Omega'})$ and positive constant c_2 such that

$$\Delta f - \frac{l}{l-1} \frac{|\nabla f|^2}{f} \geq -c_2 \text{ in } \Omega', \quad f(x) > 0 \text{ in } \Omega', \quad f(x) = 0 \text{ on } \partial\Omega'. \quad (4.9)$$

Now we compare $u(x, t)$ with

$$w(x, t) = c_3 (f(x) + c_2(T - t))^{-1/(l-1)}$$

in $\Omega' \times (0, T)$, where the positive constant c_3 will be defined below. By (4.9) for $x \in \Omega'$ and $t \in [0, T)$ we get

$$w_t - \Delta w + c(x, t)w^p \geq \frac{w}{(l-1)[f(x) + c_2(T - t)]} \left(c_2 + \Delta f - \frac{l|\nabla f|^2}{(l-1)[f(x) + c_2(T - t)]} \right) \geq 0.$$

Choosing c_3 such that $c_3 \geq c_2^{1/(l-1)} c_1$ and $w(x, 0) \geq u_0(x)$ for $x \in \Omega'$, by comparison principle we conclude

$$u(x, t) \leq w(x, t) \text{ in } \overline{\Omega'} \times [0, T).$$

Hence, $u(x, t)$ cannot blow up in $\Omega' \times [0, T]$. Since Ω' is an arbitrary subset of Ω , the proof is completed. \square

From [27] it is easy to get the following result.

Theorem 4.3. *Let $p > 1$, $\inf_{Q_T} c(x, t) > 0$ and the solution of (1.1)–(1.3) blows up in finite time. Then blow-up occurs only on the boundary.*

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